An International Journal
computers & mathematics with applications

PERGAMON Computers and Mathematics with Applications 45 (2003) 1075–1085
www.elsevier.nl/locate/camwa

Existence and Uniqueness of Solutions to Discrete Diffusion Equations

D. R. ANDERSON
Department of Mathematics and Computer Science, Concordia College
Moorhead, MN 56562, U.S.A.
andersod@cord.edu

R. I. AVERY
College of Natural Sciences, Dakota State University
Madison, SD 57042, U.S.A.
averyr@pluto.dsu.edu

J. M. DAVIS
Department of Mathematics, Baylor University
Waco, TX 76798, U.S.A.
John_M_Davis@baylor.edu

Abstract—Motivated by the classical one-dimensional diffusion equation,

$$\begin{aligned} u_t - ku_{xx} &= f(x,t), && x \in \mathbb{R}, \quad t > 0, \\ u(x,0) &= \psi(x), && x \in \mathbb{R}, \end{aligned}$$

we find the unique solution of the partial difference equation

$$\begin{aligned} \Delta_t u(x,t) - k\Delta^2_{x,x} u(x-1,t) &= f(x,t), && x \in \mathbb{Z}, \quad t \geq 0, \\ u(x,0) &= \psi(x), && x \in \mathbb{Z}, \end{aligned}$$

where $k \in (0,1)$. Our approach is to construct the Green's function for the appropriate partial difference operator, and almost immediately we establish an existence and uniqueness result for the discrete nonhomogeneous problem.

Keywords—Partial difference equations, Green's function, Discrete diffusion equation.

1. INTRODUCTION

The existence and uniqueness of solutions of the (continuous) diffusion equation

$$\begin{aligned} u_t - ku_{xx} &= f(x,t), && x \in \mathbb{R}, \quad t > 0, \\ u(x,0) &= \psi(x), && x \in \mathbb{R}, \end{aligned}$$

with $k \in (0,1)$ is a classical result discussed in elementary partial differential equations texts;

0898-1221/03/$ - see front matter Typeset by $\mathcal{A}_{\mathcal{M}}\mathcal{S}$-TEX
PII: S0898-1221(03)00087-7

see, for example, [1]. The unique solution is given by

$$u(x,t) = \int_{-\infty}^{\infty} G_c(x,y,t,0)\psi(y)\,dy + \int_0^t \int_{-\infty}^{\infty} G_c(x,y,t,s)g(y,s)\,dy\,ds,$$

where

$$G_c(x,y,t,s) = \frac{1}{2\sqrt{\pi k(t-s)}} \exp\left(\frac{-(x-y)^2}{4k(t-s)}\right)$$

is the Green's function for the (continuous) diffusion equation. In this paper, we find the Green's function for the discrete diffusion equation which has similar properties to the Green's function for the (continuous) diffusion equation, then conclude with an existence-uniqueness theorem for the discrete diffusion equation. The existence-uniqueness theorem and the form of the solution for the discrete diffusion equation resembles the existence-uniqueness theorem and the form of the solution for the (continuous) diffusion equation. The proofs are based on first principles of difference equations. Kelley and Peterson [2] and Agarwal [3] are recommended references on the subject.

There are very few results in partial difference equations. The entire theory of partial difference equations has been described as being an underdeveloped research area [4]. In [5], de Boor *et al.* have used Fourier series arguments to conclude the existence of fundamental solutions to partial difference equations with constant coefficients. Veit [6] followed by giving conditions for the existence of fundamental solutions to partial difference equations with nonconstant coefficients. Existence-uniqueness results are noted in their work, again using Fourier series arguments. In this work, we will find a Green's function for the discrete diffusion equation, highlight some of the important properties of the Green's function, and then use elementary difference equations techniques to prove the existence and uniqueness of solutions to nonhomogeneous (nonfinite support) discrete diffusion equations.

2. PRELIMINARIES

In this section, we will find a Green's function for the operator

$$L_{x,t}u(x,t) = \Delta_t u(x,t) - k\Delta_{x,x}^2 u(x-1,t),$$

where $u : \mathbb{Z} \times \mathbb{W} \to \mathbb{R}$ and $k \in (0,1)$ and then explain the properties of the Green's function we will use in our main results. Note that throughout the paper we represent the set of integers by $\mathbb{Z}$ and the set of whole numbers (nonnegative integers) by $\mathbb{W}$. We begin by verifying that the function S defined by

$$S(x,t) = \begin{cases} 1 + \sum\limits_{i=x+1}^{t} \begin{pmatrix} 2i-1 \\ i-x-1 \end{pmatrix} \begin{pmatrix} t \\ t-i \end{pmatrix} (-1)^{i-x} k^i, & \text{for } x \geq 0, \\ 1 - S(-x-1,t), & \text{for } x < 0, \end{cases}$$

is a solution of

$$L_{x,t}S(x,t) = 0, \qquad \text{for all } x \in \mathbb{Z} \text{ and } t \in \mathbb{W},$$

with

$$S(x,0) = \begin{cases} 1, & \text{if } x \geq 0, \\ 0, & \text{if } x < 0. \end{cases}$$

The proof of the following theorem highlights the properties of the function S, which will be used to define the Green's function for the discrete diffusion equation. The proof relies on Pascal's formula and the convention that if the lower limit of summation is greater than the upper limit of summation, then the sum is zero.

THEOREM 2.1. *The function S satisfies the following properties:*

(1) *S is defined for $(x,t) \in \mathbb{Z} \times \mathbb{W}$;*
(2) *$S(x,0) = 1$, for all $x \geq 0$ and $t \in \mathbb{W}$;*
(3) *$S(x,0) = 0$ for all $x < 0$ and $t \in \mathbb{W}$; and*
(4) *$L_{x,t}S(x,t) = 0$, for all $(x,t) \in \mathbb{Z} \times \mathbb{W}$.*

PROOF. By the definition of S, Properties 1–3 are clearly satisfied. We verify Property 4 by exhaustion. Let $(x,t) \in \mathbb{Z} \times \mathbb{W}$.

CASE 1. $x > 0$.

$$\begin{aligned}
L_{x,t}S(x,t) &= S(x,t+1) - kS(x+1,t) + (2k-1)S(x,t) - kS(x-1,t) \\
&= 1 + \sum_{i=x+1}^{t+1} \binom{2i-1}{i-x-1}\binom{t+1}{t+1-i}(-1)^{i-x}k^i \\
&\quad - k\left(1 + \sum_{i=x+2}^{t} \binom{2i-1}{i-x-2}\binom{t}{t-i}(-1)^{i-x-1}k^i\right) \\
&\quad + (2k-1)\left(1 + \sum_{i=x+1}^{t} \binom{2i-1}{i-x-1}\binom{t}{t-i}(-1)^{i-x}k^i\right) \\
&\quad - k\left(1 + \sum_{i=x}^{t} \binom{2i-1}{i-x}\binom{t}{t-i}(-1)^{i-x+1}k^i\right) \\
&= \sum_{i=x+1}^{t+1} \binom{2i-1}{i-x-1}\binom{t+1}{t+1-i}(-1)^{i-x}k^i \\
&\quad - k\sum_{i=x+2}^{t} \binom{2i-1}{i-x-2}\binom{t}{t-i}(-1)^{i-x-1}k^i \\
&\quad + (2k-1)\sum_{i=x+1}^{t} \binom{2i-1}{i-x-1}\binom{t}{t-i}(-1)^{i-x}k^i \\
&\quad - k\sum_{i=x}^{t} \binom{2i-1}{i-x}\binom{t}{t-i}(-1)^{i-x+1}k^i \\
&= \sum_{i=x+1}^{t} \binom{2i-1}{i-x-1}\left[\binom{t+1}{t+1-i} - \binom{t}{t-i}\right](-1)^{i-x}k^i \\
&\quad + k\sum_{i=x+2}^{t} \left[\binom{2i-1}{i-x-2} + \binom{2i-1}{i-x-1}\right]\binom{t}{t-i}(-1)^{i-x}k^i \\
&\quad + k\sum_{i=x+1}^{t} \left[\binom{2i-1}{i-x-1} + \binom{2i-1}{i-x}\right]\binom{t}{t-i}(-1)^{i-x}k^i \\
&\quad + \binom{2t+1}{t-x}\binom{t+1}{0}(-1)^{t+1-x}k^{t+1} \\
&\quad - \binom{2x+1}{0}\binom{t}{t-x-1}k^{x+2} \\
&\quad + \binom{2x-1}{0}\binom{t}{t-x}k^{x+1} \\
&= \sum_{i=x+1}^{t} \binom{2i-1}{i-x-1}\binom{t}{t+1-i}(-1)^{i-x}k^i
\end{aligned}$$

$$+k\sum_{i=x+2}^{t}\binom{2i}{i-x-1}\binom{t}{t-i}(-1)^{i-x}k^{i}$$
$$+k\sum_{i=x+1}^{t}\binom{2i}{i-x}\binom{t}{t-i}(-1)^{i-x}k^{i}$$
$$+\binom{2t+1}{t-x}\binom{t+1}{0}(-1)^{t+1-x}k^{t+1}$$
$$-\binom{2x+1}{0}\binom{t}{t-x-1}k^{x+2}$$
$$+\binom{2x-1}{0}\binom{t}{t-x}k^{x+1}$$
$$=\sum_{i=x+1}^{t}\binom{2i-1}{i-x-1}\binom{t}{t+1-i}(-1)^{i-x}k^{i}$$
$$+k\sum_{i=x+2}^{t}\binom{2i+1}{i-x}\binom{t}{t-i}(-1)^{i-x}k^{i}$$
$$+\binom{2t+1}{t-x}\binom{t+1}{0}(-1)^{t+1-x}k^{t+1}$$
$$-\binom{2x+1}{0}\binom{t}{t-x-1}k^{x+2}$$
$$+\binom{2x-1}{0}\binom{t}{t-x}k^{x+1}$$
$$-\binom{2x+2}{1}\binom{t}{t-x-1}k^{x+2}$$
$$=\sum_{i=x+1}^{t}\binom{2i-1}{i-x-1}\binom{t}{t+1-i}(-1)^{i-x}k^{i}$$
$$+k\sum_{i=x+3}^{t+1}\binom{2i-1}{i-x-1}\binom{t}{t-i+1}(-1)^{i-x-1}k^{i-1}$$
$$+\binom{2t+1}{t-x}\binom{t+1}{0}(-1)^{t+1-x}k^{t+1}$$
$$-\binom{2x+1}{0}\binom{t}{t-x-1}k^{x+2}$$
$$+\binom{2x-1}{0}\binom{t}{t-x}k^{x+1}$$
$$-\binom{2x+2}{1}\binom{t}{t-x-1}k^{x+2}$$
$$=\sum_{i=x+3}^{t}\binom{2i-1}{i-x-1}\binom{t}{t+1-i}(-1)^{i-x}k^{i}$$
$$+\sum_{i=x+3}^{t}\binom{2i-1}{i-x-1}\binom{t}{t-i+1}(-1)^{i-x-1}k^{i}$$
$$+\binom{2t+1}{t-x}\binom{t+1}{0}(-1)^{t+1-x}k^{t+1}$$
$$-\binom{2x+1}{0}\binom{t}{t-x-1}k^{x+2}$$

$$
\begin{aligned}
&+\binom{2x-1}{0}\binom{t}{t-x}k^{x+1} \\
&-\binom{2x+2}{1}\binom{t}{t-x-1}k^{x+2} \\
&-\binom{2x+1}{0}\binom{t}{t-x}k^{x+1} \\
&+\binom{2x+3}{1}\binom{t}{t-x-1}k^{x+2} \\
&+\binom{2t+1}{t-x}\binom{t}{0}(-1)^{t-x}k^{t+1} \\
=&-\binom{2x+1}{0}\binom{t}{t-x-1}k^{x+2} \\
&-\binom{2x+2}{1}\binom{t}{t-x-1}k^{x+2} \\
&+\binom{2x+3}{1}\binom{t}{t-x-1}k^{x+2} \\
=&-\binom{2x+1}{0}\binom{t}{t-x-1}k^{x+2} \\
&+\binom{2x+2}{0}\binom{t}{t-x-1}k^{x+2} \\
=&\,0.
\end{aligned}
$$

CASE 2. $x = 0$.

Since

$$\binom{2i-1}{i-1} = \binom{2i-1}{i}$$

for all integers $i \geq 1$, we have

$$S(-1,t) = -\sum_{i=1}^{t}\binom{2i-1}{i-1}\binom{t}{t-i}(-1)^i k^i = \sum_{i=1}^{t}\binom{2i-1}{i}\binom{t}{t-i}(-1)^{i+1}k^i.$$

Therefore,

$$
\begin{aligned}
L_{x,t}S(x,t) &= S(0,t+1) - kS(1,t) + (2k-1)S(0,t) - kS(-1,t) \\
&= S(0,t+1) - kS(1,t) + (2k-1)S(0,t) - k(1 - S(0,t)) \\
&= 1 + \sum_{i=1}^{t+1}\binom{2i-1}{i-1}\binom{t+1}{t+1-i}(-1)^i k^i \\
&\quad - k\left(1 + \sum_{i=2}^{t}\binom{2i-1}{i-2}\binom{t}{t-i}(-1)^{i-1}k^i\right) \\
&\quad + (2k-1)\left(1 + \sum_{i=1}^{t}\binom{2i-1}{i-1}\binom{t}{t-i}(-1)^i k^i\right) \\
&\quad - k\sum_{i=1}^{t}\binom{2i-1}{i}\binom{t}{t-i}(-1)^{i+1}k^i \\
&= k + \sum_{i=1}^{t+1}\binom{2i-1}{i-1}\binom{t+1}{t+1-i}(-1)^i k^i \\
&\quad - k\sum_{i=2}^{t}\binom{2i-1}{i-2}\binom{t}{t-i}(-1)^{i-1}k^i
\end{aligned}
$$

$$
\begin{aligned}
&+(2k-1)\sum_{i=1}^{t}\binom{2i-1}{i-1}\binom{t}{t-i}(-1)^i k^i \\
&-k\sum_{i=1}^{t}\binom{2i-1}{i}\binom{t}{t-i}(-1)^{i+1}k^i \\
&= k+\sum_{i=1}^{t}\binom{2i-1}{i-1}\left[\binom{t+1}{t+1-i}-\binom{t}{t-i}\right](-1)^i k^i \\
&-k\sum_{i=2}^{t}\binom{2i-1}{i-2}\binom{t}{t-i}(-1)^{i-1}k^i \\
&+2k\sum_{i=1}^{t}\binom{2i-1}{i-1}\binom{t}{t-i}(-1)^i k^i \\
&-k\sum_{i=1}^{t}\binom{2i-1}{i}\binom{t}{t-i}(-1)^{i+1}k^i \\
&+\binom{2t+1}{t}\binom{t+1}{0}(-1)^{t+1}k^{t+1} \\
&= k+\sum_{i=1}^{t}\binom{2i-1}{i-1}\left[\binom{t+1}{t+1-i}-\binom{t}{t-i}\right](-1)^i k^i \\
&+k\sum_{i=2}^{t}\left[\binom{2i-1}{i-2}+\binom{2i-1}{i-1}\right]\binom{t}{t-i}(-1)^i k^i \\
&+k\sum_{i=1}^{t}\left[\binom{2i-1}{i-1}+\binom{2i-1}{i}\right]\binom{t}{t-i}(-1)^i k^i \\
&+\binom{2t+1}{t}\binom{t+1}{0}(-1)^{t+1}k^{t+1} \\
&-\binom{1}{0}\binom{t}{t-1}k^2 \\
&= k+\sum_{i=1}^{t}\binom{2i-1}{i-1}\binom{t}{t+1-i}(-1)^i k^i \\
&+k\sum_{i=2}^{t}\binom{2i}{i-1}\binom{t}{t-i}(-1)^i k^i \\
&+k\sum_{i=1}^{t}\binom{2i}{i}\binom{t}{t-i}(-1)^i k^i \\
&+\binom{2t+1}{t}\binom{t+1}{0}(-1)^{t+1}k^{t+1} \\
&-\binom{1}{0}\binom{t}{t-1}k^2 \\
&= k+\sum_{i=1}^{t}\binom{2i-1}{i-1}\binom{t}{t+1-i}(-1)^i k^i \\
&+k\sum_{i=2}^{t}\left[\binom{2i}{i-1}+\binom{2i}{i}\right]\binom{t}{t-i}(-1)^i k^i \\
&+\binom{2t+1}{t}\binom{t+1}{0}(-1)^{t+1}k^{t+1}
\end{aligned}
$$

$$
\begin{aligned}
& - \binom{1}{0}\binom{t}{t-1}k^2 \\
& - \binom{2}{1}\binom{t}{t-1}k^2 \\
&= k + \sum_{i=1}^{t}\binom{2i-1}{i-1}\binom{t}{t+1-i}(-1)^i k^i \\
&\quad + \sum_{i=3}^{t+1}\binom{2i-1}{i-1}\binom{t}{t-i+1}(-1)^{i-1}k^i \\
&\quad + \binom{2t+1}{t}\binom{t+1}{0}(-1)^{t+1}k^{t+1} \\
&\quad - \binom{1}{0}\binom{t}{t-1}k^2 \\
&\quad - \binom{2}{1}\binom{t}{t-1}k^2 \\
&= k + \sum_{i=3}^{t}\binom{2i-1}{i-1}\binom{t}{t+1-i}(-1)^i k^i \\
&\quad + \sum_{i=3}^{t}\binom{2i-1}{i-1}\binom{t}{t-i+1}(-1)^{i-1}k^i \\
&\quad + \binom{2t+1}{t}\binom{t+1}{0}(-1)^{t+1}k^{t+1} \\
&\quad - \binom{1}{0}\binom{t}{t-1}k^2 \\
&\quad - \binom{2}{1}\binom{t}{t-1}k^2 \\
&\quad + \binom{2t+1}{t}\binom{t}{0}(-1)^{t+1-1}k^{t+1} \\
&\quad - \binom{1}{0}\binom{t}{t}k \\
&\quad + \binom{3}{1}\binom{t}{t-1}k^2 \\
&= 0.
\end{aligned}
$$

CASE 3. $x < 0$.

By definition of S, we have $S(x,t) = 1 - S(-x-1,t)$, and hence, by Case 1 if $x \le -2$, we have

$$
\begin{aligned}
L_{x,t}S(x,t) &= S(x,t+1) - kS(x+1,t) + (2k-1)S(x,t) - kS(x-1,t) \\
&= (1 - S(-x-1,t+1)) - k(1 - S(-x-2,t)) \\
&\quad + (2k-1)(1 - S(-x-1,t)) - k(1 - S(-x,t)) \\
&= -[S(-x-1,t+1) - kS(-x,t)) + (2k-1)S(-x-1,t) - kS(-x-2,t)] \\
&= -L_{x,t}S(-x-1,t) = 0,
\end{aligned}
$$

and by Case 2, if $x = -1$, we have

$$
\begin{aligned}
L_{x,t}S(-1,t) &= S(-1,t+1)) - kS(0,t) + (2k-1)S(-1,t) - kS(-2,t) \\
&= (1 - S(0,t+1)) - kS(0,t) \\
&\quad + (2k-1)(1 - S(0,t)) - k(1 - S(1,t)) \\
&= -[S(0,t+1) - kS(1,t) + (2k-1)S(0,t) - k(1 - S(0,t))] \\
&= -L_{x,t}S(0,t) = 0.
\end{aligned}
$$

Therefore, for all $(x,t) \in \mathbb{Z} \times \mathbb{W}$ we have

$$L_{x,t}S(x,t) = 0. \qquad \blacksquare$$

In the following lemma, we verify the properties of S required in our main results. The Green's function will then be defined accordingly.

LEMMA 2.1. *For all $x, y \in \mathbb{Z}$ and $t, s \in \mathbb{W}$ with $t > s$:*

(1) $L_{x,t}S(y-x-1, t-s-1) = 0$; *and*
(2) $L_{x,t}\Delta_y S(y-x-1, t-s-1) = 0$.

PROOF. Suppose $x, y \in \mathbb{Z}$ and $t, s \in \mathbb{W}$ with $t > s$. Let $X = y-x-1$, $Z = y-x$, and $T = t-s-1$; then $X, Z \in \mathbb{Z}$ and $T \in \mathbb{W}$. Therefore, by Theorem 2.1,

$$\begin{aligned} L_{x,t}S(y-x-1,t-s-1) &= S(y-x-1,t-s) - kS(y-x-2,t-s-1) \\ &\quad + (2k-1)S(y-x-1,t-s-1) - kS(y-x,t-s-1) \\ &= S(X,T+1) - kS(X-1,T) \\ &\quad + (2k-1)S(X,T) - kS(X+1,T) \\ &= L_{X,T}S(X,T) \\ &= 0, \end{aligned}$$

and thus,

$$\begin{aligned} L_{x,t}\Delta_y S(y-x-1,t-s-1) &= L_{x,t}[S(y-x,t-s-1) - S(y-x-1,t-s-1)] \\ &= L_{x,t}S(y-x,t-s-1) - L_{x,t}S(y-x-1,t-s-1) \\ &= L_{Z,T}S(Z,T) - L_{X,T}S(X,T) \\ &= 0. \end{aligned} \qquad \blacksquare$$

Define the function

$$G(x,y,t,s) = \Delta_y S(y-x-1,t-s-1)$$

for all $x, y \in \mathbb{Z}$ and $t, s \in \mathbb{W}$ with $t > s$. The function G is the Green's function for the discrete diffusion equation. In Lemma 2.1, we have verified that for all $x, y \in \mathbb{Z}$ and $t, s \in \mathbb{W}$ with $t > s$,

$$L_{x,t}G(x,y,t,s) = 0,$$

and similarly one can show that

$$L_{x,t}G(x,y,t+1,s) = 0,$$

for all $x, y \in \mathbb{Z}$ and $t, s \in \mathbb{W}$ with $t \geq s$. The important properties of the Green's function are inherited from S and were proven in Theorem 2.1 and Lemma 2.1.

3. EXISTENCE AND UNIQUENESS OF SOLUTIONS

We begin by finding a solution to the homogeneous discrete diffusion equation with initial condition at $t = 0$. Note that this solution has the same form as the solution of the homogeneous diffusion equation with initial condition at $t = 0$.

LEMMA 3.1. *If*

$$f : \mathbb{Z} \to \mathbb{R}, \qquad \text{where } \lim_{x \to \infty} f(x) = 0,$$

then

$$v(x,t) = \sum_{y=-\infty}^{\infty} G(x,y,t+1,0) f(y)$$

is a solution of the homogeneous discrete diffusion equation

$$L_{x,t}v(x,t) = 0, \qquad \text{with } v(x,0) = f(x).$$

PROOF. Let $x \in \mathbb{Z}$ and $t \in \mathbb{W}$. Then by the note to Lemma 2.1,

$$\begin{aligned} L_{x,t}v(x,t) &= L_{x,t} \sum_{y=-\infty}^{\infty} G(x,y,t+1,0)f(y) \\ &= \sum_{y=-\infty}^{\infty} L_{x,t}G(x,y,t+1,0)f(y) \\ &= \sum_{y=-\infty}^{\infty} 0 \cdot f(y) \\ &= 0, \end{aligned}$$

and by summation by parts, since $\lim_{r\to\infty f(r)=0}$, we have

$$\begin{aligned} v(x,t) &= \sum_{y=-\infty}^{\infty} G(x,y,t+1,0)f(y) \\ &= \sum_{y=-\infty}^{\infty} \Delta_y S(y-x-1,t)f(y) \\ &= \lim_{r\to\infty} S(r-x-1,t)f(r) - \lim_{r\to-\infty} S(r-x-1,t)f(r) \\ &\quad - \sum_{y=-\infty}^{\infty} S(y-x,t)\Delta_y f(y) \\ &= \lim_{r\to\infty} 1 \cdot f(r) - \lim_{r\to-\infty} 0 \cdot f(r) \\ &\quad - \sum_{y=-\infty}^{\infty} S(y-x,t)\Delta_y f(y) \\ &= - \sum_{y=-\infty}^{\infty} S(y-x,t)\Delta_y f(y). \end{aligned}$$

Therefore, again since $\lim_{r\to\infty} f(r) = 0$, we have

$$\begin{aligned} v(x,0) &= - \sum_{y=-\infty}^{\infty} S(y-x,0)\Delta_y f(y) \\ &= - \sum_{y=x}^{\infty} \Delta_y f(y) \\ &= - \lim_{r\to\infty} f(r) + f(x) \\ &= f(x). \end{aligned}$$

∎

Next, we find the solution of the nonhomogeneous discrete diffusion equation with zero initial condition at $t = 0$. Note that this solution has the same form as the solution of the nonhomogeneous (continuous) diffusion equation with zero initial condition at $t = 0$.

LEMMA 3.2. *If*

$$g : \mathbb{Z} \times \mathbb{W} \to \mathbb{R}, \qquad \text{where } \lim_{x\to\infty} g(x,t) = 0,$$

for all $t \in \mathbb{W}$, *then*

$$w(x,t) = \sum_{s=0}^{t-1} \sum_{y=-\infty}^{\infty} G(x,y,t,s)g(y,s)$$

is a solution of the nonhomogeneous discrete diffusion equation

$$L_{x,t}w(x,t) = g(x,t), \qquad \text{with } w(x,0) = 0.$$

PROOF. By definition,

$$w(x,0) = \sum_{s=0}^{-1} \sum_{y=-\infty}^{\infty} G(x,y,0,s)g(y,s) = 0.$$

Let $x \in \mathbb{Z}$ and $t \in \mathbb{W}$. Then by Lemma 2.1 we have

$$\begin{aligned}
L_{x,t}w(x,t) &= L_{x,t} \sum_{s=0}^{t-1} \sum_{y=-\infty}^{\infty} G(x,y,t,s)g(y,s) \\
&= \sum_{y=-\infty}^{\infty} G(x,y,t+1,t)g(y,t) + \sum_{s=0}^{t-1} \sum_{y=-\infty}^{\infty} L_{x,t}G(x,y,t,s)g(y,s) \\
&= \sum_{y=-\infty}^{\infty} G(x,y,t+1,t)g(y,t) \\
&= \sum_{y=-\infty}^{\infty} \Delta_y S(y-x-1,0)g(y,t) \\
&= \lim_{r\to\infty} S(r-x-1,0)g(r,t) - \lim_{r\to-\infty} S(r-x-1,0)g(r,t) \\
&\quad - \sum_{y=-\infty}^{\infty} S(y-x,0)\Delta_y g(y,t) \\
&= - \sum_{y=-\infty}^{\infty} S(y-x,0)\Delta_y g(y,t) \\
&= - \sum_{y=x}^{\infty} \Delta_y g(y,t) \\
&= - \lim_{r\to\infty} g(y,t) + g(x,t) = g(x,t),
\end{aligned}$$

since for all $t \in \mathbb{W}$, $\lim_{r\to\infty} g(y,t) = 0$. ∎

Combining the results of Lemmas 3.1 and 3.2, we have the unique solution of the nonhomogeneous discrete diffusion equation with a initial condition at $t = 0$.

THEOREM 3.1. *The unique solution of the nonhomogeneous discrete diffusion equation*

$$L_{x,t}u(x,t) = g(x,t), \qquad \text{with } u(x,0) = f(x),$$

is given by

$$u(x,t) = \sum_{y=-\infty}^{\infty} G(x,y,t+1,0)f(y) + \sum_{s=0}^{t-1} \sum_{y=-\infty}^{\infty} G(x,y,t,s)g(y,s).$$

PROOF. By Lemmas 3.1 and 3.2, we have

$$\begin{aligned}
L_{x,t}u(x,t) &= L_{x,t} \sum_{y=-\infty}^{\infty} G(x,y,t+1,0)f(y) + L_{x,t} \sum_{s=0}^{t-1} \sum_{y=-\infty}^{\infty} G(x,y,t,s)g(y,s) \\
&= 0 + g(x,t) \\
&= g(x,t)
\end{aligned}$$

and

$$\begin{aligned} u(x,0) &= \sum_{y=-\infty}^{\infty} G(x,y,1,0)f(y) + \sum_{s=0}^{-1}\sum_{y=-\infty}^{\infty} G(x,y,0,s)g(y,s) \\ &= f(x) + 0 \\ &= f(x). \end{aligned}$$

If $z(x,t)$ is any other solution, then

$$q(x,t) = z(x,t) - u(x,t)$$

is a solution of

$$L_{x,t}q(x,t) = 0, \qquad \text{with } q(x,0) = 0,$$

and by iteration it is easily verified that

$$q(x,t) = 0,$$

for all $x \in \mathbb{Z}$ and $t \in \mathbb{W}$ is the only solution of this boundary value problem. Therefore, $u = z$ and

$$u(x,t) = \sum_{y=-\infty}^{\infty} G(x,y,t+1,0)f(y) + \sum_{s=0}^{t-1}\sum_{y=-\infty}^{\infty} G(x,y,t,s)g(y,s)$$

is the unique solution of the nonhomogeneous discrete diffusion equation

$$L_{x,t}u(x,t) = g(x,t), \qquad \text{with } u(x,0) = f(x). \qquad \blacksquare$$

REFERENCES

1. J.D. Logan, *Applied Partial Differential Equations*, Springer, New York, (1998).
2. W.G. Kelley and A.C. Peterson, *Difference Equations*, Second Edition, Harcourt/Academic Press, San Diego, CA, (2001).
3. R.P. Agarwal, *Difference Equations and Inequalities*, Dekker, New York, (1992).
4. S.S. Cheng, An underdeveloped research area: Qualitative theory of partial functional difference equations, In *International Mathematics Conference '94*, (Kaohsiung, 1994), pp. 65–75, World Sci., River Edge, NJ, (1996).
5. C. de Boor, K. Höllig and S. Riemenschneider, Fundamental solutions for multivariate difference equations, *Amer. J. Math.* **111** (3), 403–415, (1989).
6. J. Veit, Boundary value problems for partial difference equations, *Multidimens. Systems Signal Process.* **7** (2), 113–134, (1996).